A Self-Contained Filtered Density Function

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Advances in Turbulence Modeling
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Background & Terminology

Status \ Methodology	S-FDF	VS-FDF	PEVC-FDF
Fundamentals & Basic Flow Simulations	Colucci *et al.* (1998) Jaberi *et al.* (1999) Garrick *et al.* (2000)	Gicquel *et al.* (2002) Sheikhi *et al.* (2007) Sheikhi *et al.* (2009)	Present work
Complex Flow Applications	Drozda *et al.* (2005) Pitsch *et al.* (2005) Raman *et al.* (2005) Jones *et al.* (2007) Yilmaz *et al.* (2010)	Nik *et al.* (2010)	Future work

Exact Filtered Transport Equations

$$\frac{\partial\langle\rho\rangle_\ell}{\partial t}+\frac{\partial\langle\rho\rangle_\ell\langle u_j\rangle_L}{\partial x_j}=0$$

$$\frac{\partial\langle\rho\rangle_\ell\langle u_i\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_\ell\langle u_i\rangle_L\langle u_j\rangle_L}{\partial x_j}=-\frac{\partial\langle p\rangle_\ell}{\partial x_i}+\frac{\partial\breve{\tau}_{ij}}{\partial x_j}-\frac{\partial\langle\rho\rangle_\ell\tau_L(u_i,u_j)}{\partial x_j}$$

$$\frac{\partial\langle\rho\rangle_\ell\langle e\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_\ell\langle u_j\rangle_L\langle e\rangle_L}{\partial x_j}=-\frac{\partial\breve{q}_j}{\partial x_j}-\frac{\partial\langle\rho\rangle_\ell\tau_L(e,u_j)}{\partial x_j}+\breve{\tau}_{ij}\frac{\partial\langle u_i\rangle_L}{\partial x_j}$$

$$+\epsilon-\Pi_d-\langle p\rangle_\ell\frac{\partial\langle u_i\rangle_L}{\partial x_i}$$

$$\frac{\partial\langle\rho\rangle_\ell\langle\phi_\alpha\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_\ell\langle u_j\rangle_L\langle\phi_\alpha\rangle_L}{\partial x_j}=-\frac{\partial\breve{J}_j^\alpha}{\partial x_j}-\frac{\partial\langle\rho\rangle_\ell\tau_L(\phi_\alpha,u_j)}{\partial x_j}+\langle\rho\rangle_\ell\langle S_\alpha\rangle_L$$

PEVC-FDF

Fine grained density

$$\zeta\Big(\boldsymbol{v},\boldsymbol{\psi},\theta,\eta;\boldsymbol{u}(\mathbf{x},t),\boldsymbol{\phi}(\mathbf{x},t),e(\mathbf{x},t),p(\mathbf{x},t)\Big)=\left(\prod_{i=1}^{3}\delta\left(v_i-u_i(\mathbf{x},t)\right)\right)$$
$$\times\left(\prod_{\alpha=1}^{\sigma=N_s}\delta\left(\psi_\alpha-\phi_\alpha(\mathbf{x},t)\right)\right)\times\delta\left(\theta-e(\mathbf{x},t)\right)\times\delta\left(\eta-p(\mathbf{x},t)\right)$$

Filtered density function

$$P_L\left(\boldsymbol{v},\boldsymbol{\psi},\theta,\eta,\mathbf{x};t\right)=\int_{-\infty}^{+\infty}\rho(\mathbf{x}',t)\zeta\Big(\boldsymbol{v},\boldsymbol{\psi},\theta,\eta;\boldsymbol{u}(\mathbf{x}',t),\boldsymbol{\phi}(\mathbf{x}',t),$$
$$e(\mathbf{x}',t),p(\mathbf{x}',t)\Big)G(\mathbf{x}'-\mathbf{x})d\mathbf{x}'$$

$$\langle\rho(\mathbf{x},t)\rangle_\ell\langle Q(\mathbf{x},t)\rangle_L=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{Q}\left(\boldsymbol{v},\boldsymbol{\psi},\theta,\eta\right)P_L(\boldsymbol{v},\boldsymbol{\psi},\theta,\eta,\mathbf{x};t)d\boldsymbol{v}d\boldsymbol{\psi}d\theta d\eta$$

Exact FDF Transport Equation

$$
\begin{aligned}
\frac{\partial P_L}{\partial t} + \frac{\partial \boldsymbol{v}_j P_L}{\partial x_j} = & \frac{\partial}{\partial \boldsymbol{v}_i}\left(\left\langle \frac{1}{\rho}\frac{\partial p}{\partial x_i} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) - \frac{\partial}{\partial \boldsymbol{v}_i}\left(\left\langle \frac{1}{\rho}\frac{\partial \tau_{ij}}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) \\
& + \frac{\partial}{\partial \psi_\alpha}\left(\left\langle \frac{1}{\rho}\frac{\partial J_j^\alpha}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) + \frac{\partial}{\partial \theta}\left(\left\langle \frac{1}{\rho}\frac{\partial q_i}{\partial x_i} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) \\
& - \frac{\partial}{\partial \theta}\left(\left\langle \frac{1}{\rho}\tau_{ij}\frac{\partial u_i}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) + \frac{\partial}{\partial \theta}\left(\left\langle \frac{1}{\rho}p\frac{\partial u_j}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) \\
& + (\gamma - 1)\frac{\partial}{\partial \eta}\left(\left\langle \frac{\partial q_i}{\partial x_i} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) - (\gamma - 1)\frac{\partial}{\partial \eta}\left(\left\langle \tau_{ij}\frac{\partial u_i}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) \\
& + \gamma\frac{\partial}{\partial \eta}\left(\left\langle p\frac{\partial u_j}{\partial x_j} \middle| \boldsymbol{v}, \boldsymbol{\psi}, \theta, \eta \right\rangle_\ell P_L\right) - \frac{\partial S_\alpha(\psi) P_L}{\partial \psi_\alpha}
\end{aligned}
$$

FDF Modeling

- Lagrangian vector variables

$$\boldsymbol{Z}^+(t) = [\mathbf{X}^+(t), \boldsymbol{U}^+(t), \boldsymbol{\phi}^+(t), E^+(t), P^+(t)]$$

- Diffusion process

$$d\boldsymbol{Z}^+ = \boldsymbol{D}(\boldsymbol{Z}^+)dt + \boldsymbol{B}(\boldsymbol{Z}^+)d\boldsymbol{W}$$

Standardized Gaussian random variable

$$\boldsymbol{Z}^+(t_{k+1}) = \boldsymbol{Z}^+(t_k) + \boldsymbol{D}\left(\boldsymbol{Z}^+(t_k)\right)\Delta t + \boldsymbol{B}\left(\boldsymbol{Z}^+(t_k)\right)\Delta t^{1/2}\zeta_k$$

- Compare the corresponding Fokker-Planck equation with FDF

$$D = \ldots \quad , \quad E = \ldots$$

Stochastic Model

$$dX_i^{\,+} = U_i^+ \, dt + \sqrt{\frac{2\mu}{\langle \rho \rangle_\ell}} dW_i$$

$$\begin{aligned} dU_i^+ = & -\frac{1}{\langle \rho \rangle_\ell} \frac{\partial \langle p \rangle_\ell}{\partial x_i} dt + \frac{2}{\langle \rho \rangle_\ell} \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \langle u_i \rangle_L}{\partial x_j} \right) dt + \frac{1}{\langle \rho \rangle_\ell} \frac{\partial}{\partial x_j} \left(\mu \frac{\partial \langle u_j \rangle_L}{\partial x_i} \right) dt \\ & - \frac{2}{3} \frac{1}{\langle \rho \rangle_\ell} \frac{\partial}{\partial x_i} \left(\mu \frac{\partial \langle u_j \rangle_L}{\partial x_j} \right) dt + G_{ij} \left(U_j^+ - \langle u_j \rangle_L \right) dt + \sqrt{C_0 \frac{\epsilon}{\langle \rho \rangle_\ell}} dW_i' \\ & + \sqrt{\frac{2\mu}{\langle \rho \rangle_\ell}} \frac{\partial \langle u_i \rangle_L}{\partial x_j} dW_j \end{aligned}$$

$$d\phi_\alpha^+ = -C_\phi \omega \left(\phi_\alpha^+ - \langle \phi_\alpha \rangle_L \right) dt + S_\alpha(\phi^+) dt$$

Stochastic Model

where:

$$G_{ij} = \left(\frac{\Pi_d}{2k \langle \rho \rangle_\ell} - \Omega \left(\frac{1}{2} + \frac{3}{4} C_0 \right) \right) \delta_{ij}$$

$$\epsilon = \langle \rho \rangle_\ell C_\epsilon k^{3/2} / \Delta_L \qquad k = \tfrac{1}{2} \tau_L (u_i, u_i) \qquad \Omega = \epsilon / (\langle \rho \rangle_\ell k)$$

$$\Pi_d = C_\Pi \left(\left\langle \langle p \rangle_\ell \frac{\partial \langle u_i \rangle_L}{\partial x_i} \right\rangle_{\ell_2} - \langle \langle p \rangle_\ell \rangle_{\ell_1} \frac{\partial \langle \langle u_i \rangle_L \rangle_{L_2}}{\partial x_i} \right)$$

Stochastic Modeling of 'E' and 'P'

Model

$$dP^+ = P^+\left(Adt + BdW_p\right)$$

$$dE^+ = E^+\left(Cdt + DdW_p\right)$$

$$d\xi^+ = \xi^+\left(Gdt + HdW_p\right)$$

Constraints

$$P^+\xi^+ = (\gamma - 1)E^+$$

$$dE^+ = \left(-C_e\Omega(E^+ - \langle e\rangle_L) + \left(\frac{E^+(\gamma-1)\epsilon}{P^+}\right)\right)dt - P^+d\xi^+$$

(all of the above) $\Rightarrow$ C, D, G, H in terms of A, B

Fokker-Planck Equation

$$\frac{\partial F_L}{\partial t} + \frac{\partial v_i F_L}{\partial x_i} = \frac{1}{\langle\rho\rangle_\ell}\frac{\partial\langle p\rangle_\ell}{\partial x_i}\frac{\partial F_L}{\partial v_i} - \frac{2}{\langle\rho\rangle_\ell}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle u_i\rangle_L}{\partial x_j}\right)\frac{\partial F_L}{\partial v_i} - \frac{1}{\langle\rho\rangle_\ell}\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle u_j\rangle_L}{\partial x_i}\right)\frac{\partial F_L}{\partial v_i}$$
$$+ \frac{2}{3}\frac{1}{\langle\rho\rangle_\ell}\frac{\partial}{\partial x_i}\left(\mu\frac{\partial\langle u_j\rangle_L}{\partial x_j}\right)\frac{\partial F_L}{\partial v_i} - \frac{\partial\left(G_{ij}\left(v_j - \langle u_j\rangle_L\right)F_L\right)}{\partial v_i} + \frac{\partial}{\partial x_i}\left(\mu\frac{\partial(F_L/\langle\rho\rangle_\ell)}{\partial x_i}\right)$$
$$+ \frac{\partial}{\partial x_i}\left(\frac{2\mu}{\langle\rho\rangle_\ell}\frac{\partial\langle u_j\rangle_L}{\partial x_i}\frac{\partial F_L}{\partial v_j}\right) + \frac{\mu}{\langle\rho\rangle_\ell}\frac{\partial\langle u_k\rangle_L}{\partial x_j}\frac{\partial\langle u_i\rangle_L}{\partial x_j}\frac{\partial^2 F_L}{\partial v_k\partial v_i} + \frac{1}{2}C_0\frac{\epsilon}{\langle\rho\rangle_\ell}\frac{\partial^2 F_L}{\partial v_i\partial v_i}$$
$$+ C_\phi\omega\frac{\partial\left(\left(\psi_\alpha - \langle\phi_\alpha\rangle_L\right)F_L\right)}{\partial\psi_\alpha} + \frac{C_e\omega}{\gamma}\frac{\partial\left(\left(\theta - \langle e\rangle_L\right)F_L\right)}{\partial\theta} - \frac{\gamma-1}{\gamma}(\epsilon)\frac{\partial}{\partial\theta}\left(\frac{\theta}{\eta}F_L\right)$$
$$- \frac{\gamma-1}{\gamma}\frac{\partial\left(\theta A F_L\right)}{\partial\theta} + \frac{\gamma-1}{\gamma^2}\frac{\partial\left(\theta B^2 F_L\right)}{\partial\theta} - \frac{\partial\left(\eta A F_L\right)}{\partial\eta} + \frac{1}{2}\frac{(\gamma-1)^2}{\gamma^2}\frac{\partial^2\left(\theta^2 B^2 F_L\right)}{\partial\theta\partial\theta}$$
$$+ \frac{\gamma-1}{\gamma}\frac{\partial^2\left(\theta\eta B^2 F_L\right)}{\partial\theta\partial\eta} + \frac{1}{2}\frac{\partial^2\left(\eta^2 B^2 F_L\right)}{\partial\eta\partial\eta} - \frac{\partial S_\alpha(\psi)P_L}{\partial\psi_\alpha}$$

Stochastic Modeling of 'E' and 'P'

$$\frac{\partial \langle \rho \rangle_\ell \langle e \rangle_L}{\partial t} + \frac{\partial \langle \rho \rangle_\ell \langle u_j \rangle_L \langle e \rangle_L}{\partial x_j} = -\frac{\partial \breve{q}_j}{\partial x_j} - \frac{\partial \langle \rho \rangle_\ell \tau_L(e, u_j)}{\partial x_j} + \breve{\tau}_{ij} \frac{\partial \langle u_i \rangle_L}{\partial x_j}$$
$$+ \epsilon - \Pi_d - \langle p \rangle_\ell \frac{\partial \langle u_i \rangle_L}{\partial x_i}$$

$$\frac{\partial \langle \rho \rangle_l \langle e \rangle_L}{\partial t} + \frac{\partial \langle \rho \rangle_l \langle e \rangle_L \langle u_j \rangle_L}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu \frac{\partial \langle e \rangle_L}{\partial x_j}\right) - \frac{\partial \langle \rho \rangle_l \tau_L(e, u_j)}{\partial x_j}$$
$$+ \frac{\epsilon}{\gamma} + \frac{\gamma - 1}{\gamma} \int A \theta F_L \, dV \, d\psi \, d\theta \, d\eta$$

Stochastic Modeling of 'E' and 'P'

$$A = -\frac{C_e\Omega}{E^+}\left(E^+ - \langle e\rangle_L\right) + \frac{1}{P^+}\left((\gamma-1)\epsilon + \gamma\breve{\tau}_{ij}\frac{\partial\langle u_i\rangle_L}{\partial x_j}\right)$$
$$-\gamma\frac{\Pi_d}{\tau_\ell(p,p)}\left(P^+ - \langle p\rangle_\ell\right) - \gamma\frac{\partial\langle u_i\rangle_L}{\partial x_i}$$

$$B = 0$$

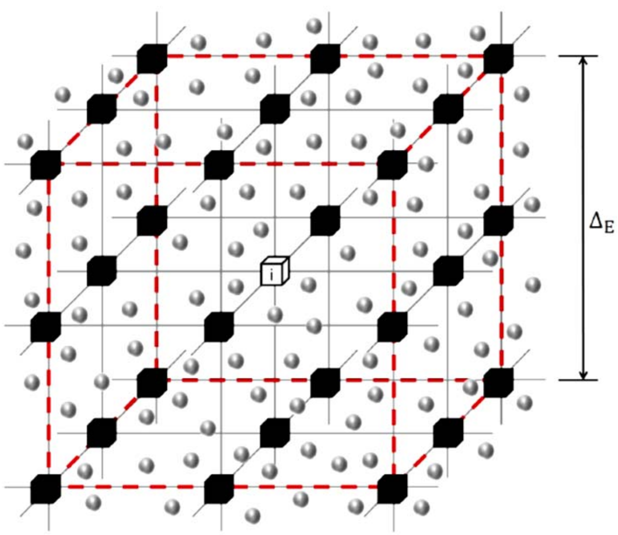
$$dP^+ = P^+\left(Adt + BdW_p\right)$$

$$dE^+ = \left(-\frac{C_e\Omega}{\gamma}\left(E^+ - \langle e\rangle_L\right) + \frac{\gamma-1}{\gamma}\frac{E^+}{P^+}\epsilon + \frac{\gamma-1}{\gamma}E^+\left(A - \frac{B^2}{\gamma}\right)\right)dt$$
$$+\frac{\gamma-1}{\gamma}E^+BdW_p$$

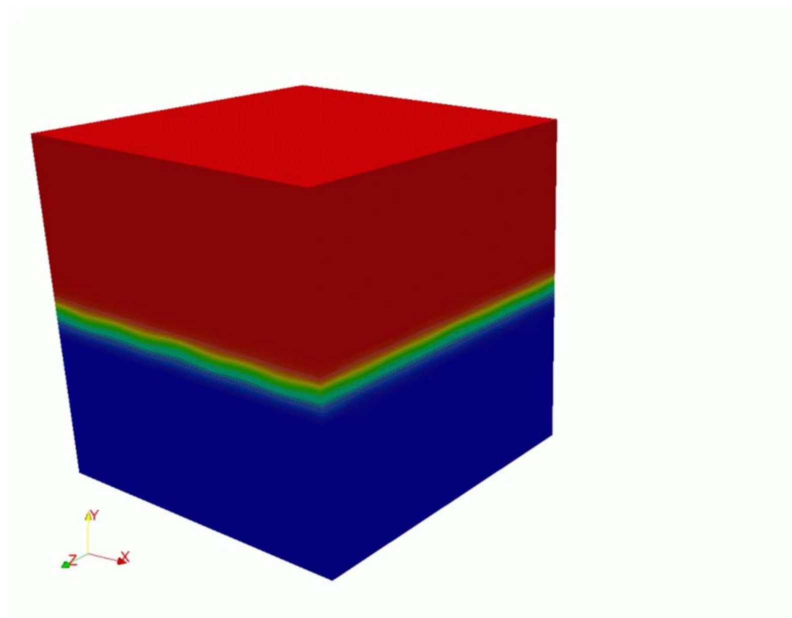
Assessment & Validation

Hybrid Eulerian-Lagrangian Methodology

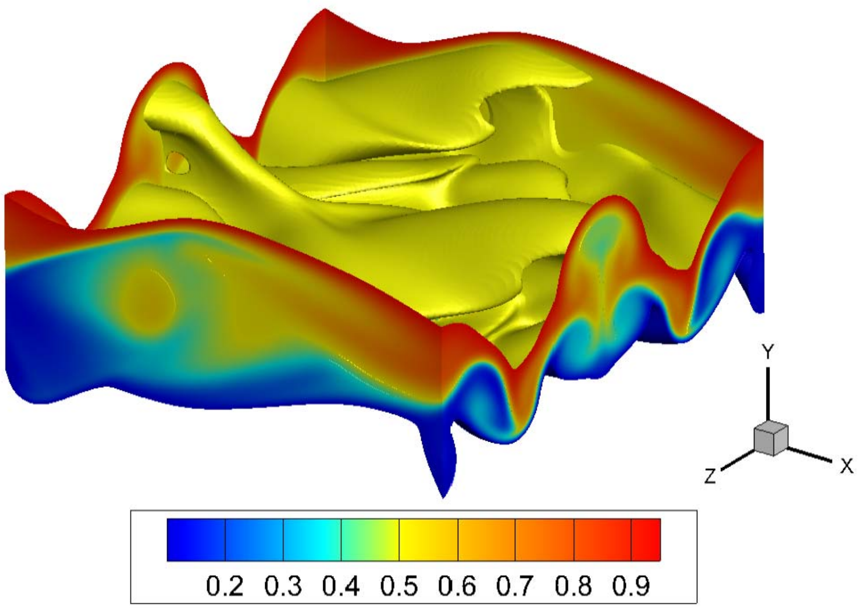
- Eulerian: Transport equations for the filtered quantities.
  - *Deterministic* simulations
- Lagrangian: SDEs
  - *Monte Carlo* simulations



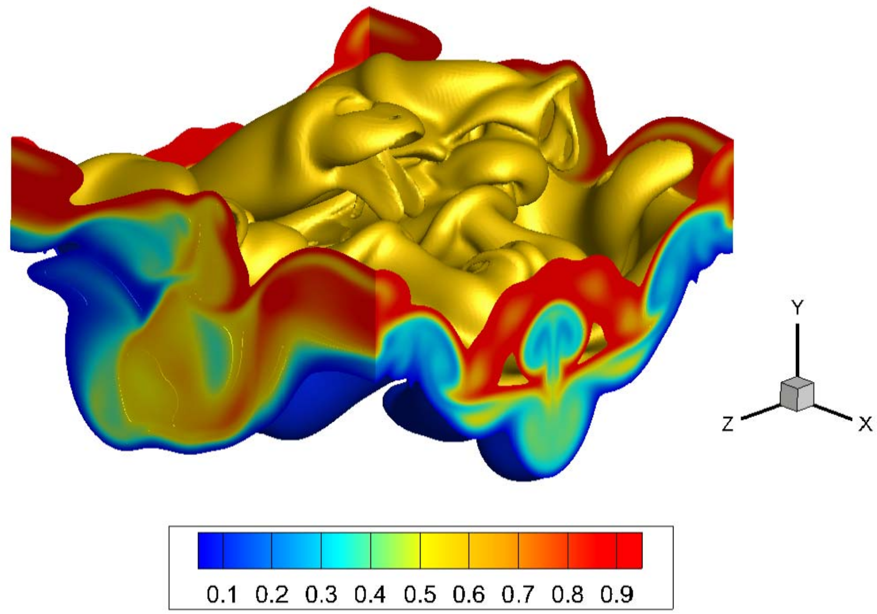
Previously:
VS-FMDF Simulation, Sheikhi *et al.*, Phys. Fluids (2009)



Passive Scalar Contour



$Ma = 1.2$



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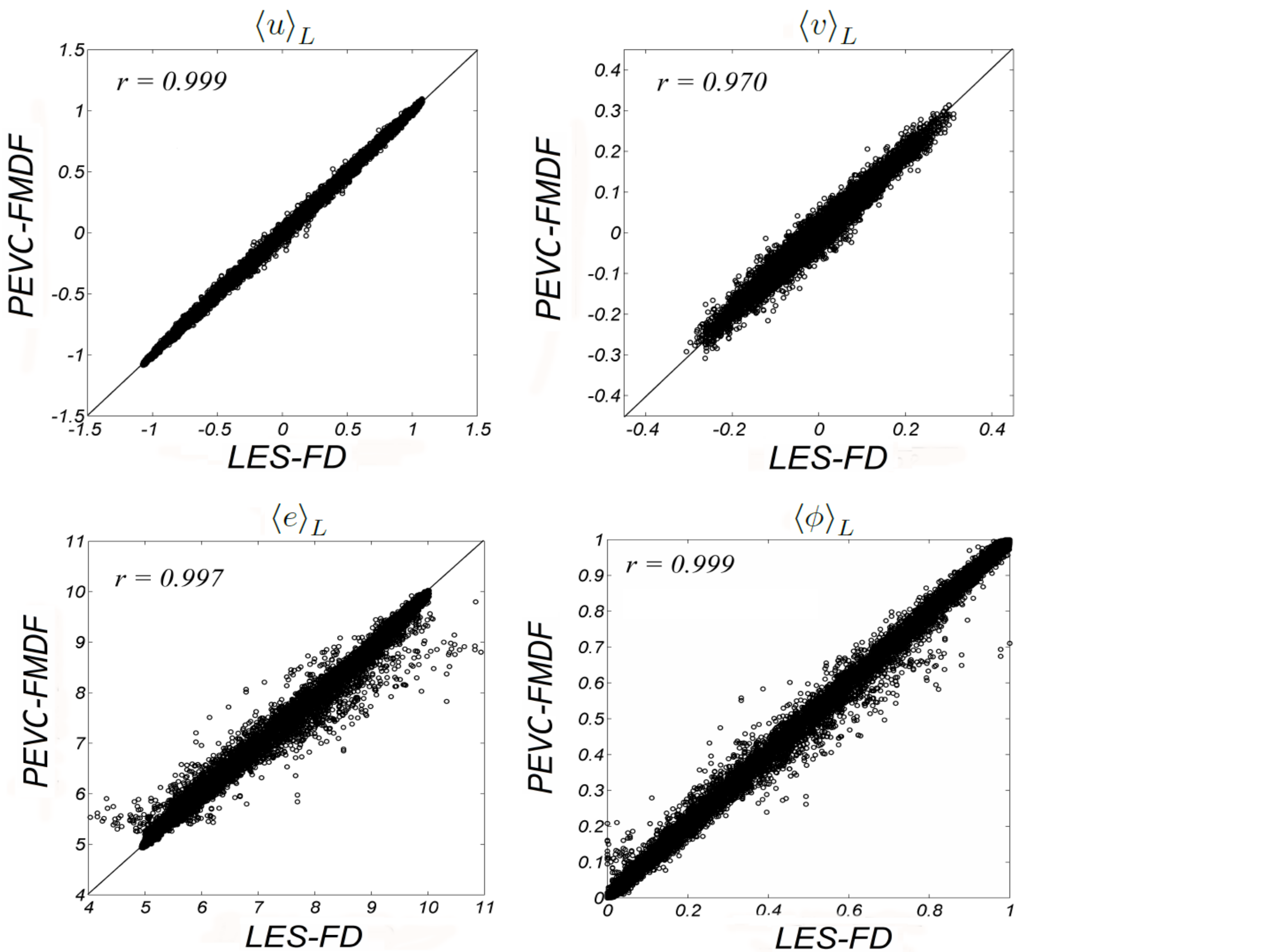
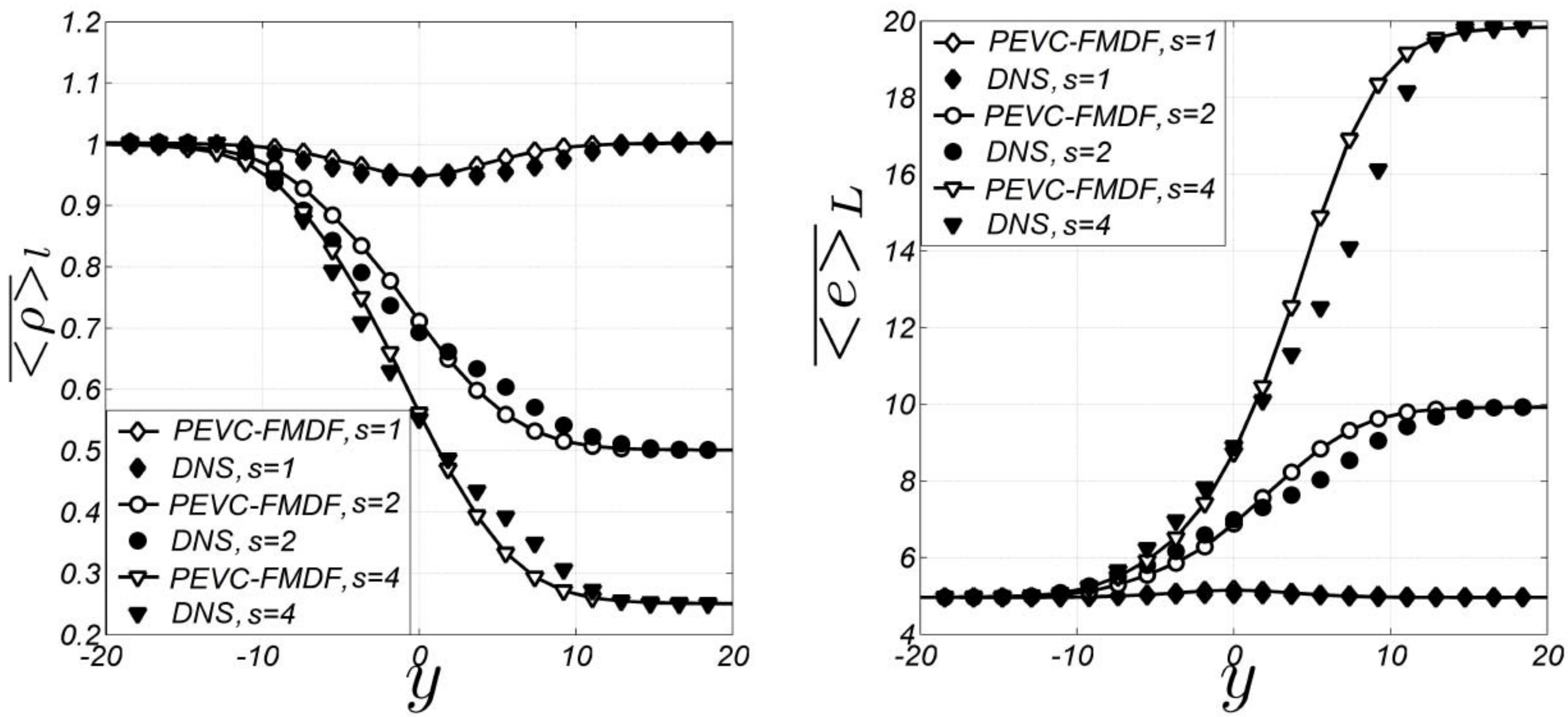
Consistency Assessment



Validation via DNS



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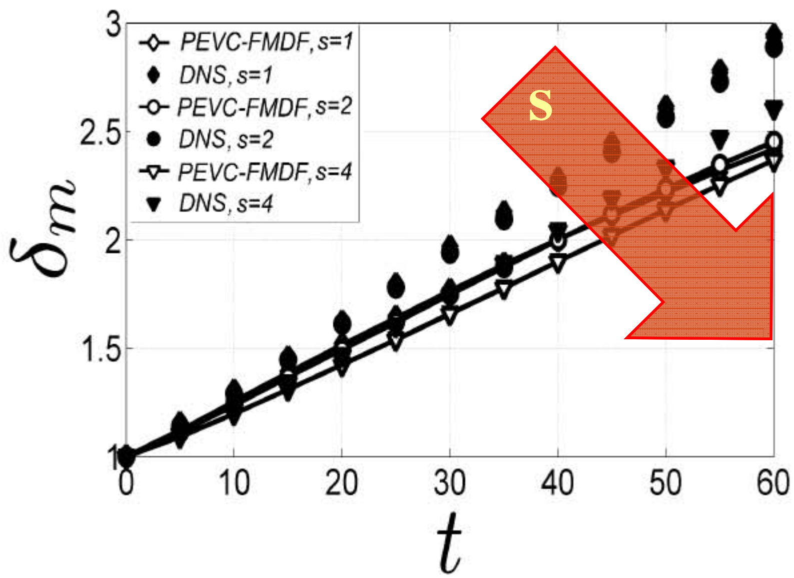
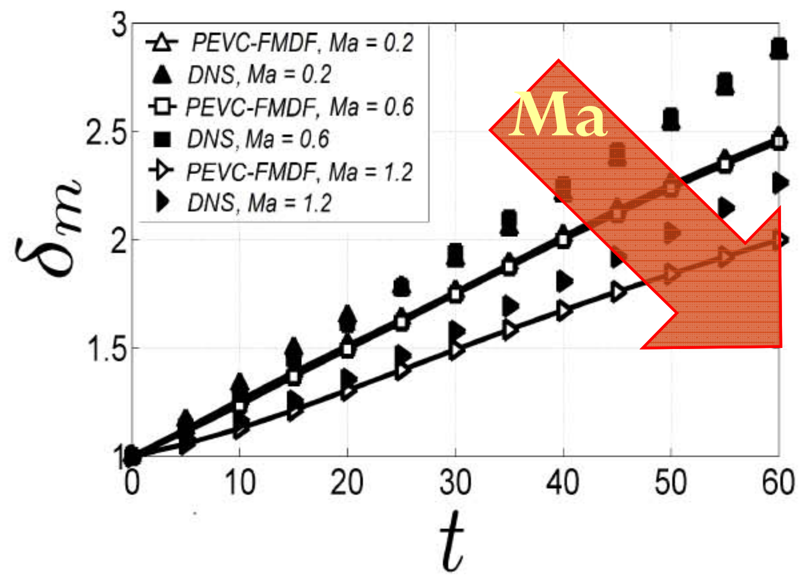
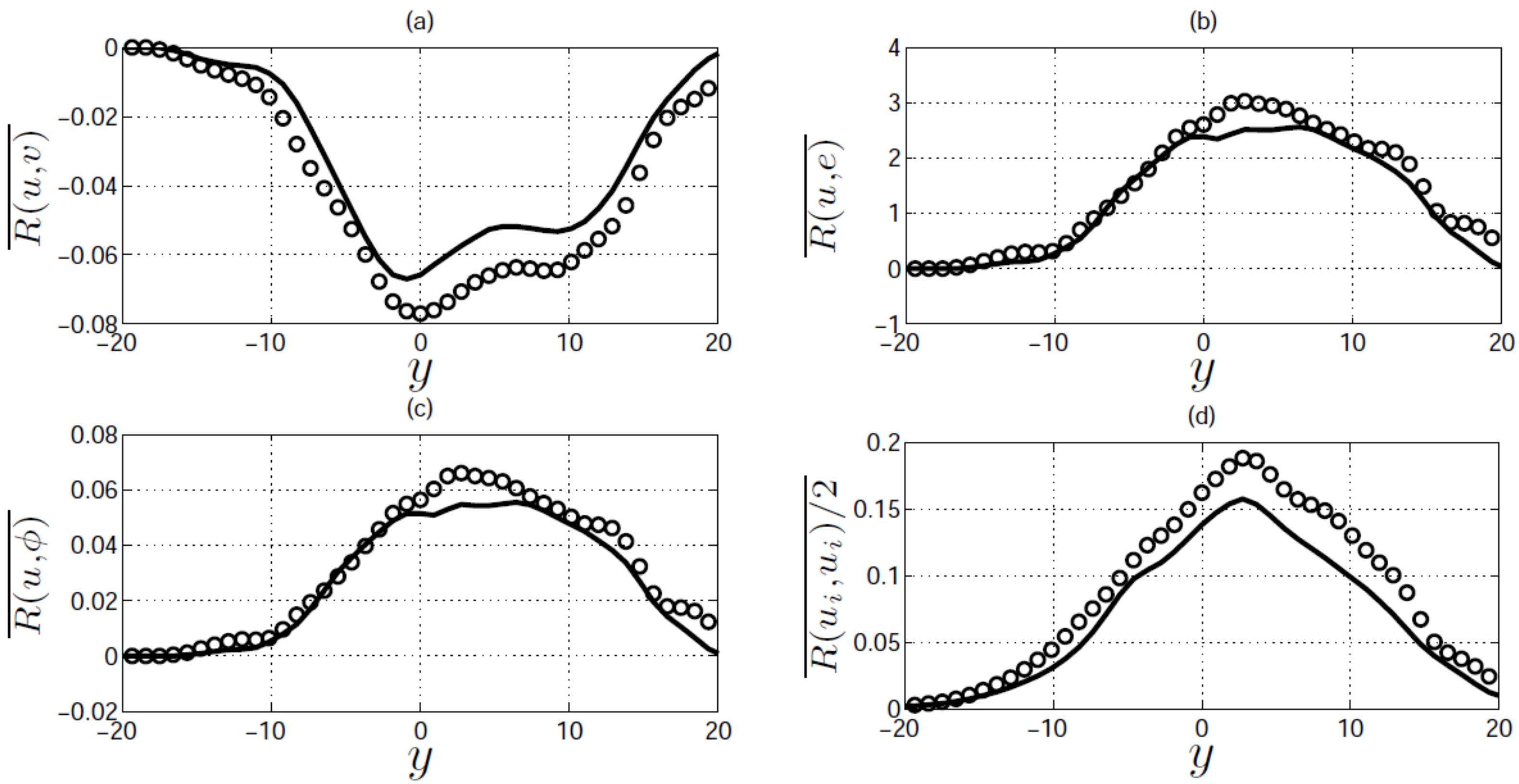




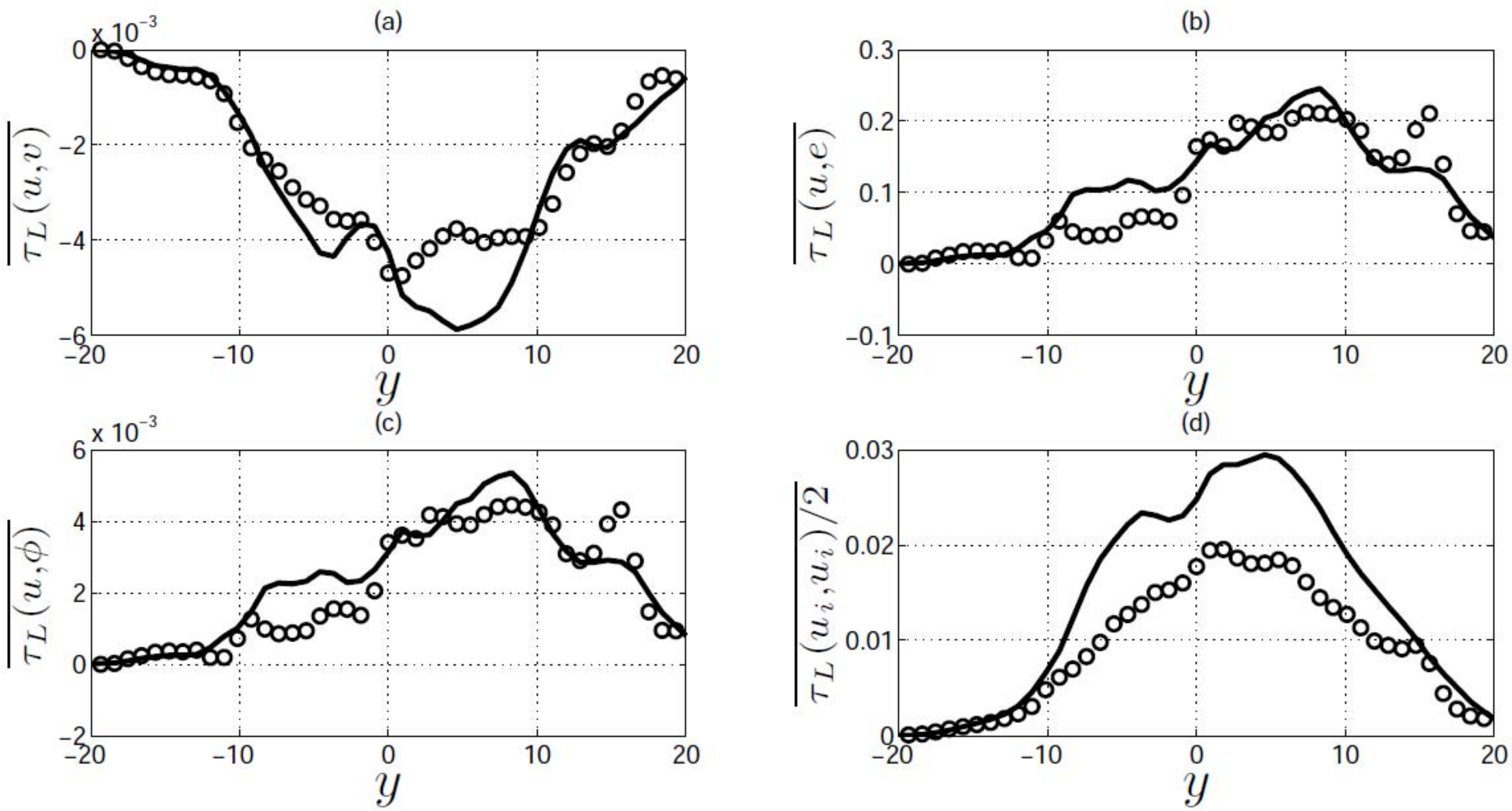
Validation via DNS

$$Ma = 0.2$$



Validation via DNS

$Ma = 0.2$



Validation via DNS

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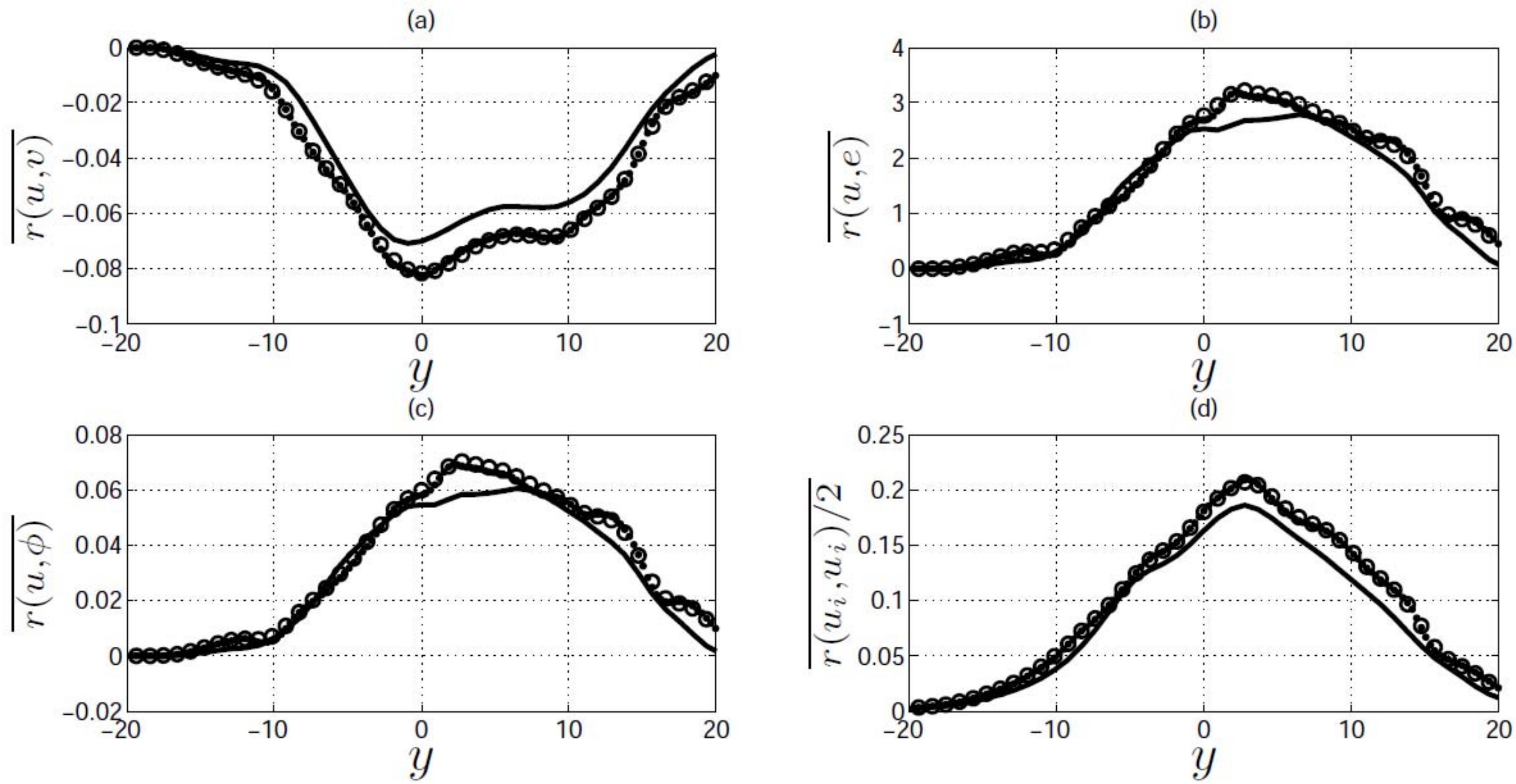
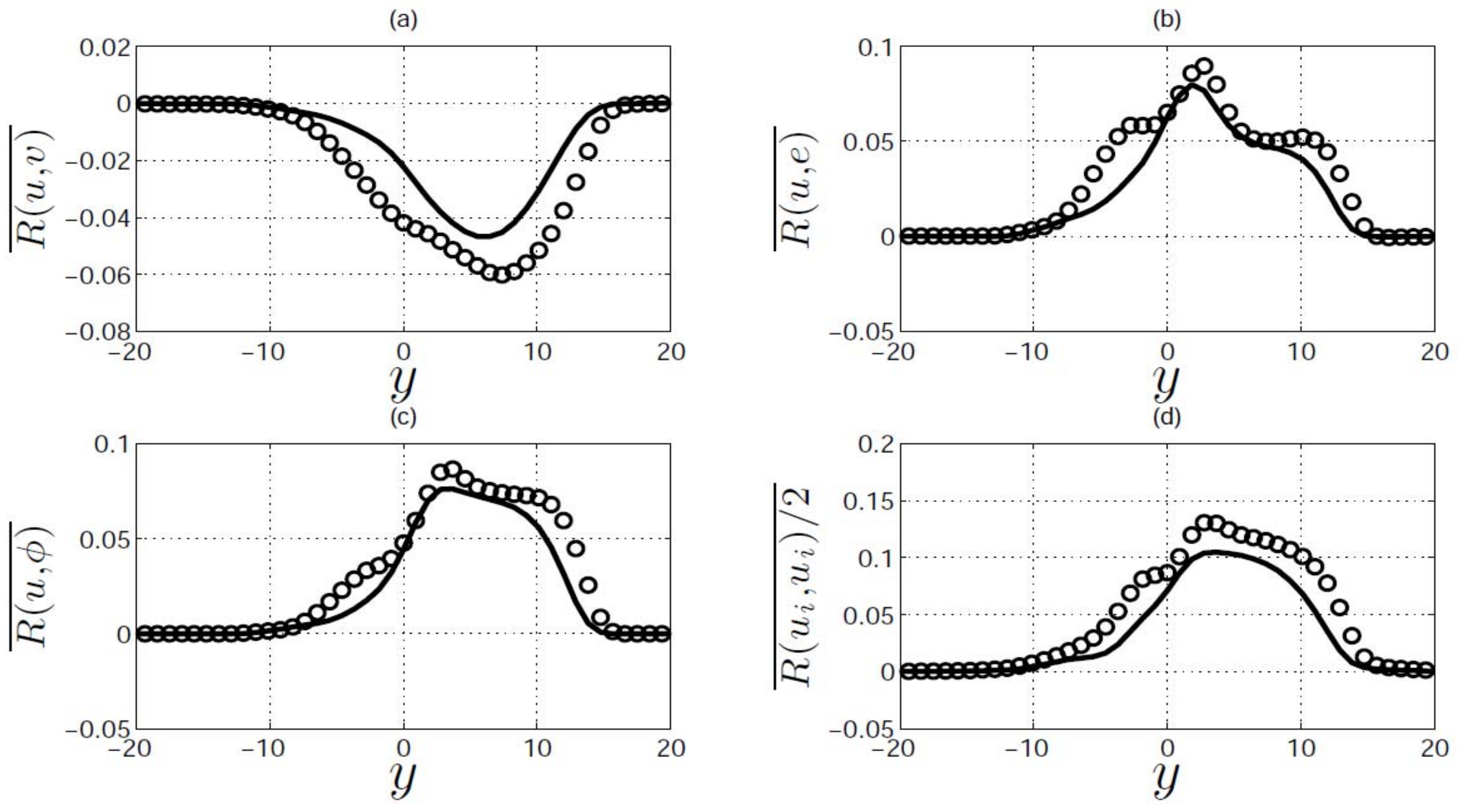


Validation via DNS

$Ma = 1.2$



Validation via DNS

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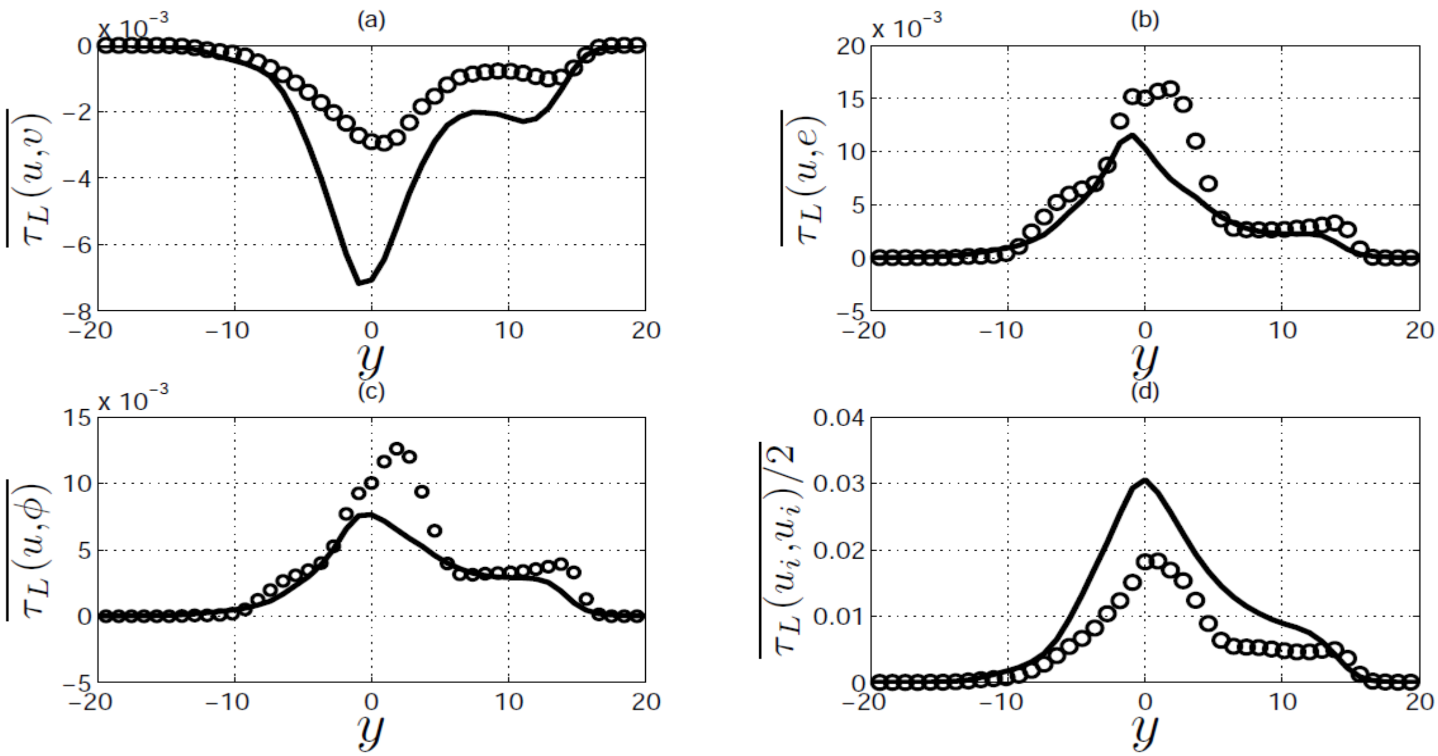
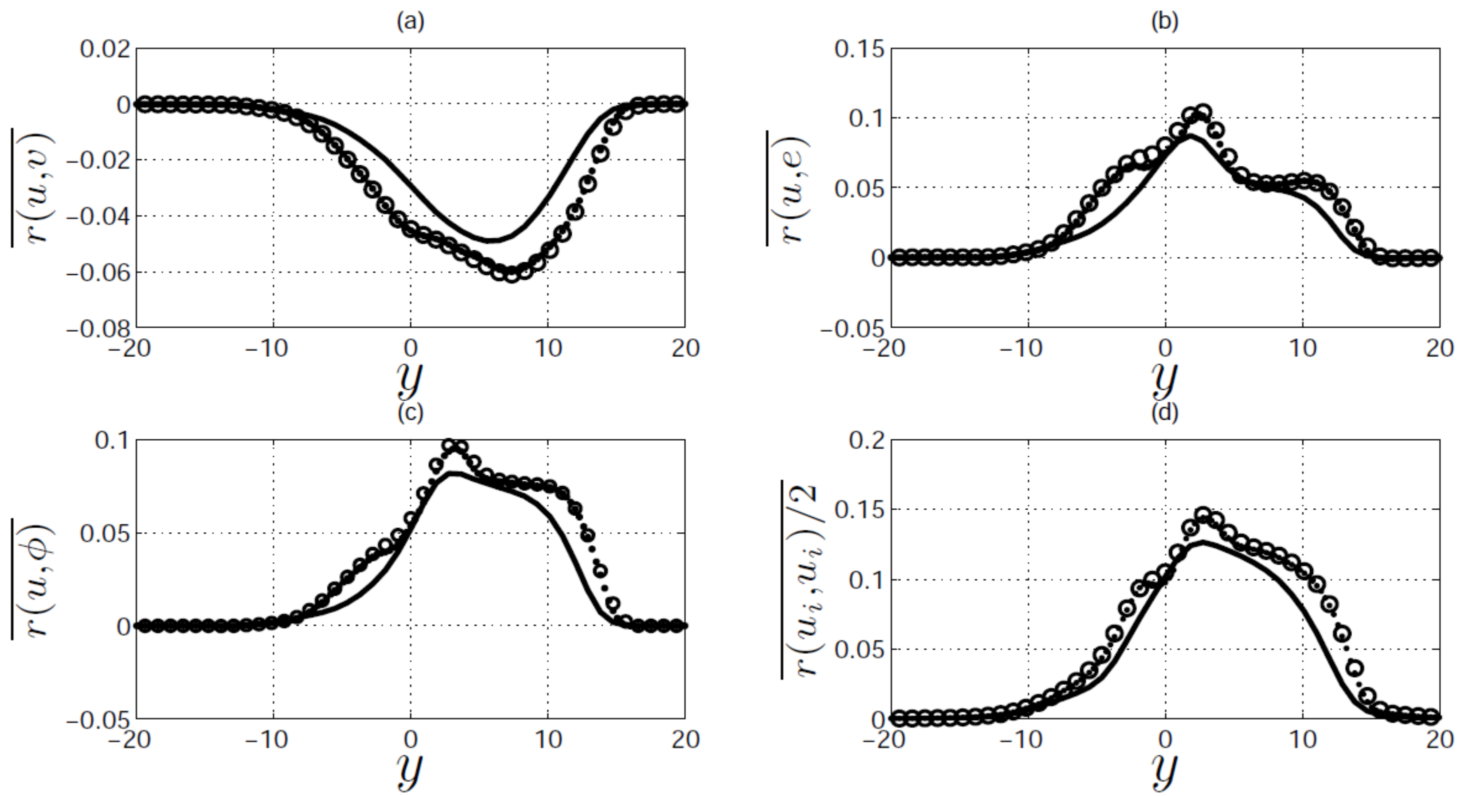


Validation via DNS

$$Ma = 1.2$$



Summary

- A self-contained FDF
  - ❖ Including the SGS statistics of all of hydro-thermo-chemical var.
  - ❖ For both high and low compressibility limits.

Future Works

- New models for pressure-strains
- New Langevin model for demonstrating RDT

- Implement on massively parallel/higher order solvers
- Complex geometries

Thank You

Definitions

$$\tau_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij} \right)$$

$$q_j = -\lambda \frac{\partial T}{\partial x_j}$$

$$J_j^\alpha = -\rho \Gamma_\alpha \frac{\partial \phi_\alpha}{\partial x_j}$$

$$Pr = \frac{c_v \mu}{\lambda}$$

$$\sigma_{ij} = \tau_{ij} - p\delta_{ij}$$

$$Sc = \frac{\mu}{\rho \Gamma}$$

Definitions

$$\breve{\tau}_{ij} = \mu \left(\frac{\partial \langle u_i \rangle_L}{\partial x_j} + \frac{\partial \langle u_j \rangle_L}{\partial x_i} - \frac{2}{3} \frac{\partial \langle u_k \rangle_L}{\partial x_k} \delta_{ij} \right)$$

$$\breve{q}_j = -\lambda \frac{\partial \langle T \rangle_L}{\partial x_j}$$

$$\breve{J}_j^{\alpha} = - \langle \rho \rangle_\ell \Gamma \frac{\partial \langle \phi_\alpha \rangle_L}{\partial x_j}$$

$$\Pi_d = \left\langle p \frac{\partial u_i}{\partial x_i} \right\rangle_\ell - \langle p \rangle_\ell \frac{\partial \langle u_i \rangle_L}{\partial x_i}$$

$$\epsilon = \left\langle \tau_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle_\ell - \breve{\tau}_{ij} \frac{\partial \langle u_i \rangle_L}{\partial x_j}$$

Transport Equation for Pressure

$$p = \rho R^0 T / \overline{M} = \rho R T = (\gamma - 1)\rho e$$

$$\frac{\partial p}{\partial t} + \frac{\partial p u_j}{\partial x_j} = -(\gamma - 1)\frac{\partial q_j}{\partial x_j} + (\gamma - 1)\,\sigma_{ij}\frac{\partial u_i}{\partial x_j}$$

Exact Second Order Correlations

$$\frac{\partial \langle\rho\rangle_\ell \tau_L(u_i,u_j)}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_k\rangle_L \tau_L(u_i,u_j)}{\partial x_k} = \langle\rho\rangle_\ell \left(-\tau_L(u_i,u_k)\frac{\partial \langle u_j\rangle_L}{\partial x_k} - \tau_L(u_j,u_k)\frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)$$
$$- \frac{\partial}{\partial x_k}\Bigg(\langle\rho\rangle_\ell \tau_L(u_i,u_j,u_k) + \tau(p,u_i)\delta_{jk} + \tau(p,u_j)\delta_{ik} - \Big(\tau(u_i,\tau_{jk})$$
$$+ \tau\left(u_j,\tau_{ik}\right)\Big)\Bigg) - \left[\left(\left\langle \tau_{ik}\frac{\partial u_j}{\partial x_k}\right\rangle_\ell - \breve{\tau}_{ik}\frac{\partial \langle u_j\rangle_L}{\partial x_k}\right) + \left(\left\langle \tau_{jk}\frac{\partial u_i}{\partial x_k}\right\rangle_\ell - \breve{\tau}_{jk}\frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)\right]$$
$$+ \left[\left(\left\langle p\frac{\partial u_i}{\partial x_j}\right\rangle_\ell - \langle p\rangle_\ell \frac{\partial \langle u_i\rangle_L}{\partial x_j}\right) + \left(\left\langle p\frac{\partial u_j}{\partial x_i}\right\rangle_\ell - \langle p\rangle_\ell \frac{\partial \langle u_j\rangle_L}{\partial x_i}\right)\right]$$

$$\frac{\partial \langle\rho\rangle_\ell \tau_L(\phi_\alpha,\phi_\beta)}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_k\rangle_L \tau_L(\phi_\alpha,\phi_\beta)}{\partial x_k} = \langle\rho\rangle_\ell \left(-\tau_L(\phi_\alpha,u_k)\frac{\partial \langle \phi_\beta\rangle_L}{\partial x_k} - \tau_L(\phi_\beta,u_k)\frac{\partial \langle \phi_\alpha\rangle_L}{\partial x_k}\right)$$
$$- \frac{\partial}{\partial x_k}\left[\langle\rho\rangle_\ell \tau_L(\phi_\alpha,\phi_\beta,u_k) - \langle\rho\rangle_\ell \Gamma \frac{\partial \tau_L(\phi_\alpha,\phi_\beta)}{\partial x_k}\right]$$
$$- \left[2\langle\rho\rangle_\ell \Gamma \tau_L\left(\frac{\partial \phi_\alpha}{\partial x_j}, \frac{\partial \phi_\beta}{\partial x_j}\right)\right],$$

Exact Second Order Correlations

$$\frac{\partial \langle\rho\rangle_\ell \tau_L(u_i,\phi_\alpha)}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_k\rangle_L \tau_L(u_i,\phi_\alpha)}{\partial x_k} = \langle\rho\rangle_\ell \left(-\tau_L(u_i,u_k)\frac{\partial \langle\phi_\alpha\rangle_L}{\partial x_k} - \tau_L(\phi_\alpha,u_k)\frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)$$
$$- \frac{\partial}{\partial x_k}\Big(\langle\rho\rangle_\ell \tau_L(u_i,\phi_\alpha,u_k) + \tau(p,\phi_\alpha)\delta_{ik} - (\tau(\phi_\alpha,\tau_{ik}) - \tau(u_i,J_k^\alpha))\Big)$$
$$+ \left[-\left(\left\langle \tau_{ik}\frac{\partial \phi_\alpha}{\partial x_k}\right\rangle_\ell - \breve{\tau}_{ik}\frac{\partial \langle\phi_\alpha\rangle_L}{\partial x_k}\right) + \left(\left\langle J_k^\alpha \frac{\partial u_i}{\partial x_k}\right\rangle_\ell - \breve{J}_k^\alpha \frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)\right.$$
$$\left. + \left(\left\langle p\frac{\partial \phi_\alpha}{\partial x_i}\right\rangle_\ell - \langle p\rangle_\ell \frac{\partial \langle\phi_\alpha\rangle_L}{\partial x_i}\right)\right]$$

$$\frac{\partial \langle\rho\rangle_\ell \tau_L(u_i,e)}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_k\rangle_L \tau_L(u_i,e)}{\partial x_k} = \langle\rho\rangle_\ell \left(-\tau_L(u_i,u_k)\frac{\partial \langle e\rangle_L}{\partial x_k} - \tau_L(e,u_k)\frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)$$
$$- \frac{\partial}{\partial x_k}\Big(\langle\rho\rangle_\ell \tau_L(u_i,e,u_k) + \tau(p,e)\delta_{ik} - (\tau(e,\tau_{ik}) - \tau(u_i,q_k))\Big)$$
$$+ \left[-\left(\left\langle \tau_{ik}\frac{\partial e}{\partial x_k}\right\rangle_\ell - \breve{\tau}_{ik}\frac{\partial \langle e\rangle_L}{\partial x_k}\right) + \left(\left\langle q_k \frac{\partial u_i}{\partial x_k}\right\rangle_\ell - \breve{q}_k \frac{\partial \langle u_i\rangle_L}{\partial x_k}\right)\right.$$
$$+ \left(\left\langle p\frac{\partial e}{\partial x_i}\right\rangle_\ell - \langle p\rangle_\ell \frac{\partial \langle e\rangle_L}{\partial x_i}\right) + \left(\left\langle u_i \tau_{kj}\frac{\partial u_k}{\partial x_j}\right\rangle_\ell - \langle u_i\rangle_L \left\langle \tau_{kj}\frac{\partial u_k}{\partial x_j}\right\rangle_\ell\right)$$
$$\left. - \left(\left\langle u_i p\frac{\partial u_j}{\partial x_j}\right\rangle_\ell - \langle u_i\rangle_L \left\langle p\frac{\partial u_j}{\partial x_j}\right\rangle_\ell\right)\right],$$

Exact Second Order Correlations

$$\frac{\partial \langle\rho\rangle_\ell \frac{\tau_L(e,e)}{2}}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_k\rangle_L \frac{\tau_L(e,e)}{2}}{\partial x_k} = \langle\rho\rangle_\ell \left(-\tau_L(e,u_k) \frac{\partial \langle e\rangle_L}{\partial x_k} \right) - \frac{1}{2}\frac{\partial}{\partial x_k}\Big(\langle\rho\rangle_\ell \tau_L(e,e,u_k)$$

$$+ 2\tau(e,q_k) \Big) + \left[\left(\left\langle q_k \frac{\partial e}{\partial x_k} \right\rangle_\ell - \breve{q}_k \frac{\partial \langle e\rangle_L}{\partial x_k} \right) \right.$$

$$+ \left(\left\langle e\tau_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle_\ell - \langle e\rangle_L \left\langle \tau_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle_\ell \right)$$

$$\left. - \left(\left\langle ep \frac{\partial u_j}{\partial x_j} \right\rangle_\ell - \langle e\rangle_L \left\langle p \frac{\partial u_j}{\partial x_j} \right\rangle_\ell \right) \right]$$

$$\frac{\partial \frac{\tau_\ell(p,p)}{2}}{\partial t} + \langle u_k\rangle_L \frac{\partial \frac{\tau_\ell(p,p)}{2}}{\partial x_k} = -\tau(p,u_k) \frac{\partial \langle p\rangle_\ell}{\partial x_k} - \frac{1}{2}\frac{\partial}{\partial x_k}\Big(\tau(p,p,u_k) + 2(\gamma-1)\tau(p,q_k) \Big)$$

$$+ (\gamma-1)\left(\left\langle q_k \frac{\partial p}{\partial x_k} \right\rangle_\ell - \breve{q}_k \frac{\partial \langle p\rangle_\ell}{\partial x_k} \right) - \gamma \langle p\rangle_\ell \Pi_d - \gamma\tau_\ell(p,p) \frac{\partial \langle u_j\rangle_L}{\partial x_j}$$

$$- \frac{2\gamma-1}{2}\tau\left(p, p, \frac{\partial u_i}{\partial x_i} \right) + (\gamma-1)\left(\left\langle \tau_{ij} p \frac{\partial u_i}{\partial x_j} \right\rangle_\ell - \langle p\rangle_\ell \left\langle \tau_{ij} \frac{\partial u_i}{\partial x_j} \right\rangle_\ell \right)$$

Fined Grained Density Transport Eq.

$$\frac{\partial \zeta}{\partial t} = -\left(\frac{\partial u_k}{\partial t}\frac{\partial \zeta}{\partial v_k} + \frac{\partial \phi_\alpha}{\partial t}\frac{\partial \zeta}{\partial \psi_\alpha} + \frac{\partial e}{\partial t}\frac{\partial \zeta}{\partial \theta} + \frac{\partial p}{\partial t}\frac{\partial \zeta}{\partial \eta}\right)$$

$$\frac{\partial \rho \zeta}{\partial t} + \frac{\partial \rho u_j \zeta}{\partial x_j} = \left(\frac{\partial p}{\partial x_j} - \frac{\partial \tau_{kj}}{\partial x_k}\right)\frac{\partial \zeta}{\partial v_j} + \left(\frac{\partial J_j^\alpha}{\partial x_j}\right)\frac{\partial \zeta}{\partial \psi_\alpha} + \left(\gamma \rho p \frac{\partial u_j}{\partial x_j} + (\gamma - 1)\,\rho \frac{\partial q_i}{\partial x_i}\right.$$
$$\left. - (\gamma - 1)\,\rho \tau_{ij}\frac{\partial u_i}{\partial x_j}\right)\frac{\partial \zeta}{\partial \eta} + \left(\frac{\partial q_i}{\partial x_i} - \tau_{ij}\frac{\partial u_i}{\partial x_j} + p\frac{\partial u_j}{\partial x_j}\right)\frac{\partial \zeta}{\partial \theta}$$

Modeled Equations

$$\frac{\partial\langle\rho\rangle_l}{\partial t}+\frac{\partial\langle\rho\rangle_l\langle u_j\rangle_L}{\partial x_j}=0$$

$$\frac{\partial\langle\rho\rangle_l\langle u_i\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_l\langle u_i\rangle_L\langle u_j\rangle_L}{\partial x_j}=-\frac{\partial\langle p\rangle_l}{\partial x_i}+\frac{\partial\widetilde{\tau_{ij}}}{\partial x_j}-\frac{\partial\langle\rho\rangle_l\tau_L(u_i,u_j)}{\partial x_j}$$

$$\frac{\partial\langle\rho\rangle_l\langle e\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_l\langle e\rangle_L\langle u_j\rangle_L}{\partial x_j}$$
$$=\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle e\rangle_L}{\partial x_j}\right)-\frac{\partial\langle\rho\rangle_l\tau_L(e,u_j)}{\partial x_j}+\frac{\epsilon}{\gamma}+\frac{\gamma-1}{\gamma}\int A\theta F_L\,dV\,d\psi\,d\theta\,d\eta$$

$$\frac{\partial\langle\rho\rangle_l\langle\phi_\alpha\rangle_L}{\partial t}+\frac{\partial\langle\rho\rangle_l\langle\phi_\alpha\rangle_L\langle u_j\rangle_L}{\partial x_j}=\frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle\phi_\alpha\rangle_L}{\partial x_j}\right)-\frac{\partial\langle\rho\rangle_l\tau_L(\phi_\alpha,u_j)}{\partial x_j}+\langle\rho\rangle_l\langle S_\alpha(\phi)\rangle_L$$

Exact and Modeled Moments

$$\frac{\partial \langle\rho\rangle_\ell \langle e\rangle_L}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_j\rangle_L \langle e\rangle_L}{\partial x_j} = -\frac{\partial \breve{q}_j}{\partial x_j} - \frac{\partial \langle\rho\rangle_\ell \tau_L(e,u_j)}{\partial x_j} + \breve{\tau}_{ij}\frac{\partial \langle u_i\rangle_L}{\partial x_j}$$

$$+ \epsilon - \Pi_d - \langle p\rangle_\ell \frac{\partial \langle u_i\rangle_L}{\partial x_i}$$

$$\frac{\partial\langle\rho\rangle_l\langle e\rangle_L}{\partial t} + \frac{\partial\langle\rho\rangle_l\langle e\rangle_L\langle u_j\rangle_L}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle e\rangle_L}{\partial x_j}\right) - \frac{\partial\langle\rho\rangle_l\tau_L(e,u_j)}{\partial x_j}$$

$$+\frac{\epsilon}{\gamma} + \frac{\gamma-1}{\gamma}\int A\theta F_L dV d\psi d\theta d\eta$$

First & Second Order Filteration

$$\langle Q(\mathbf{x},t)\rangle_\ell = \int_{-\infty}^{+\infty} Q(\mathbf{x}',t) G_{\Delta_1}(\mathbf{x}',\mathbf{x}) d\mathbf{x}'$$

$$\langle\langle Q(\mathbf{x},t)\rangle_\ell\rangle_{\ell_2} = \int_{-\infty}^{+\infty} \langle Q(\mathbf{x}',t)\rangle_\ell G_{\Delta_2}(\mathbf{x}',\mathbf{x}) d\mathbf{x}'$$

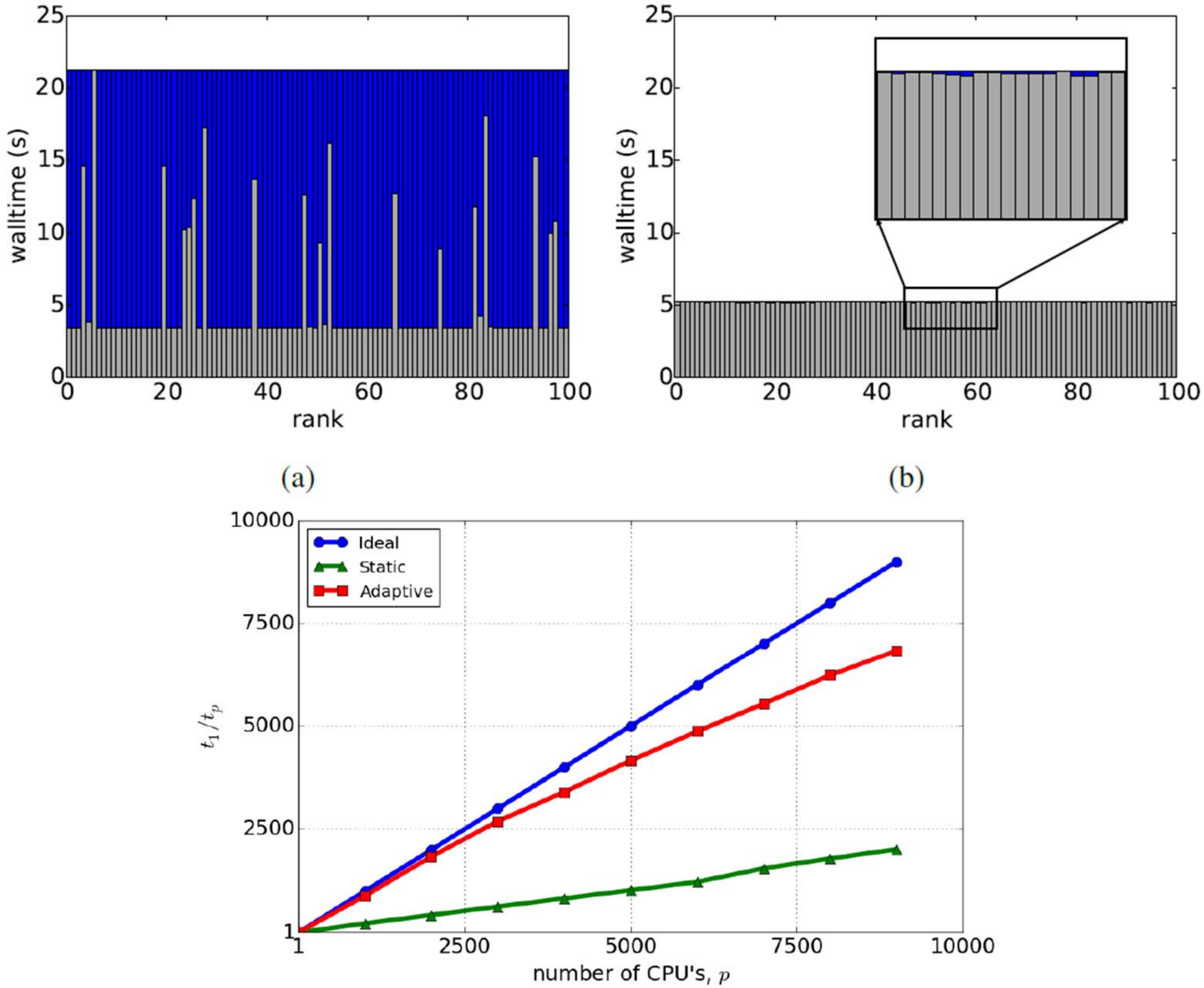
$$\langle\langle Q(\mathbf{x},t)\rangle_L\rangle_{L_2} = \langle\langle \rho Q\rangle_\ell\rangle_{\ell_2} \,/\, \langle\langle \rho\rangle_\ell\rangle_{\ell_2}$$

Stochastic Modeling of 'E' and 'P'

$$\frac{\partial \langle\rho\rangle_\ell \langle e\rangle_L}{\partial t} + \frac{\partial \langle\rho\rangle_\ell \langle u_j\rangle_L \langle e\rangle_L}{\partial x_j} = -\frac{\partial \breve{q}_j}{\partial x_j} - \frac{\partial \langle\rho\rangle_\ell \tau_L(e, u_j)}{\partial x_j} + \breve{\tau}_{ij}\frac{\partial \langle u_i\rangle_L}{\partial x_j} + \epsilon - \Pi_d - \langle p\rangle_\ell \frac{\partial \langle u_i\rangle_L}{\partial x_i}$$

$$\frac{\partial\langle\rho\rangle_l\langle e\rangle_L}{\partial t} + \frac{\partial\langle\rho\rangle_l\langle e\rangle_L\langle u_j\rangle_L}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\mu\frac{\partial\langle e\rangle_L}{\partial x_j}\right) - \frac{\partial\langle\rho\rangle_l\tau_L(e,u_j)}{\partial x_j} + \frac{\epsilon}{\gamma} + \frac{\gamma-1}{\gamma}\int A\theta F_L dV d\psi d\theta d\eta$$

IPLMC


walltime (s)
rank
(a)
(b)
Ideal
Static
Adaptive
t_1/t_p
number of CPU's, p